

Useful facts about square matrices

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1 Notation/Definition cheat sheet

Definition 1.1 Let $\mathbf{M}(n, \mathbb{F})$ denote the set of $n \times n$ matrices over field \mathbb{F} .

Specifically, let $\mathbf{M}(n, \mathbb{R})$ denote the set of real $n \times n$ matrices and $\mathbf{M}(n, \mathbb{C})$ the set of complex $n \times n$ matrices.

Definition 1.2 Let $I_n \in \mathbf{M}(n, \mathbb{F})$ be the matrix s.t. for all matrices A in $\mathbf{M}(n, \mathbb{F})$,

$$I_n A = A I_n = A \tag{1}$$

Call this matrix the identity matrix. When n and \mathbb{F} are unambiguous or arbitrary, denote the identity matrix as simply I .

1.1 Matrix Operations

Definition 1.3 For matrix $A \in \mathbf{M}(n, \mathbb{F})$ with elements a_{ij} , let A^T be the transpose of A defined by $[A^T]_{ij} = a_{ji}$. Equivalently,

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \quad (2)$$

Definition 1.4 For $A \in \mathbf{M}(n, \mathbb{C})$ with elements a_{ij} , define \overline{A} as the conjugate of A defined by the map

$$a_{ij} \mapsto a_{ij}^* \quad (3)$$

ie. take the complex conjugate of each element.

Definition 1.5 For $A \in \mathbf{M}(n, \mathbb{C})$, define A^\dagger as the Hermitian conjugate (or conjugate transpose) of A defined by

$$A^\dagger = (\overline{A})^T = \overline{A^T} \quad (4)$$

Equivalently defined by the map

$$a_{ij} \mapsto a_{ji}^* \quad (5)$$

Definition 1.6 For matrices $A, B \in \mathbf{M}(n, \mathbb{F})$, let $[A, B]$ be the commutator of A and B defined by

$$[A, B] = AB - BA \quad (6)$$

Similarly, let $\{A, B\}$ be the anti-commutator of A and B defined by

$$\{A, B\} = AB + BA \quad (7)$$

We say A and B commute iff

$$[A, B] = 0 \iff AB = BA \quad (8)$$

and say A and B anti-commute iff

$$\{A, B\} = 0 \iff AB = -BA \quad (9)$$

Definition 1.7 For matrix $A \in \mathbf{M}(n, \mathbb{F})$, let $\text{Tr}[A]$ be the trace of A defined by

$$\text{Tr}[A] = \sum_i^n a_{ii} \quad (10)$$

ie. the sum of diagonal elements of A .

1.2 Matrix Properties

Definition 1.8 For $A \in \mathbf{M}(n, \mathbb{R})$, say A is symmetric iff

$$A = A^T \quad (11)$$

and say A is anti-symmetric or skew-symmetric iff

$$A = -A^T \quad (12)$$

Definition 1.9 For $A \in \mathbf{M}(n, \mathbb{C})$, say A is Hermitian iff

$$A = A^\dagger \quad (13)$$

and say A is anti-Hermitian or skew-Hermitian iff

$$A = -A^\dagger \quad (14)$$

Definition 1.10 Say $A \in \mathbf{M}(n, \mathbb{F})$ with elements a_{ij} is diagonal iff $a_{ij} = 0 \forall i \neq j$. Ie. A has off-diagonal elements of only 0.

Definition 1.11 Say $A \in \mathbf{M}(n, \mathbb{F})$ is invertible iff \exists a unique matrix A^{-1} s.t.

$$AA^{-1} = A^{-1}A = I \quad (15)$$

If such an A^{-1} exists, call it the inverse of A .

Definition 1.12 Say $A \in \mathbf{M}(n, \mathbb{R})$ is orthogonal iff A is invertible and

$$A^{-1} = A^T \quad (16)$$

ie. the inverse of A is also its transpose.

Definition 1.13 For $A \in \mathbf{M}(n, \mathbb{C})$, say A is unitary iff A is invertible and

$$A^\dagger = A^{-1} \quad (17)$$

ie. the inverse of A is also its Hermitian conjugate.

Definition 1.14 For $A \in \mathbf{M}(n, \mathbb{R})$, say A is diagonalizable iff \exists orthogonal matrix S and diagonal matrix D s.t.

$$A = SDS^T \quad (18)$$

For $A \in \mathbf{M}(n, \mathbb{C})$, call A diagonalizable if \exists unitary S and diagonal D s.t.

$$A = SDS^\dagger \quad (19)$$

1.3 Vectors

We will need a notion of vectors and vector notation to describe some matrix properties.

Definition 1.15 Let an $n \times 1$ matrix be called a column vector and a $1 \times n$ matrix be called a row vector.

Definition 1.16 For real n -dimensional column vectors u and v , define $u \cdot v$ as the dot product defined as

$$u \cdot v = u^T v = \sum_i^n u_i v_i \quad (20)$$

where u_i, v_i are the i^{th} elements of u and v respectively.

Similarly, for complex n -dimensional u and v , define the dot product as

$$u \cdot v = u^\dagger v = \sum_i^n u_i^* v_i \quad (21)$$

1.4 Misc.

Sometimes, it is convenient to represent a matrix by its rows or columns:

$$A = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_n & \text{---} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{pmatrix} \quad (22)$$

where $r_i = (a_{i1}, a_{i2}, \dots, a_{in})$ and $c_j = (a_{1j}, a_{2j}, \dots, a_{nj})$. Sometimes we use the notation $a_{i*} = (a_{i1}, a_{i2}, \dots, a_{in})$ for rows and $a_{*j} = (a_{1j}, a_{2j}, \dots, a_{nj})$ for columns.

2 Operator properties

2.1 Transpose

Definition given in Defn. 1.3. For the following, assume $A, B \in \mathbf{M}(n, \mathbb{F})$ and $c \in \mathbb{F}$

Property 2.1 Matrix transpose is distributive over addition and scalar multiplication

Pf.

$$(A + cB)^T = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} + \begin{pmatrix} cb_{11} & \dots & cb_{1n} \\ \vdots & \ddots & \vdots \\ cb_{n1} & \dots & cb_{nn} \end{pmatrix}^T \quad (23)$$

$$= \begin{pmatrix} a_{11} + cb_{11} & \dots & a_{1n} + cb_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + cb_{n1} & \dots & a_{nn} + cb_{nn} \end{pmatrix}^T \quad (24)$$

$$= \begin{pmatrix} a_{11} + cb_{11} & \dots & a_{n1} + cb_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + cb_{1n} & \dots & a_{nn} + cb_{nn} \end{pmatrix} \quad (25)$$

$$= \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} + c \begin{pmatrix} b_{11} & \dots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{nn} \end{pmatrix} \quad (26)$$

$$= A^T + cB^T \quad (27)$$

Property 2.2 $(AB)^T = B^T A^T$

Pf.

$$(AB)^T = \begin{pmatrix} \dots & a_1 & \dots \\ \dots & a_2 & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_n & \dots \end{pmatrix} \begin{pmatrix} \left| \right. & \left| \right. & \dots & \left| \right. \\ b_1 & b_2 & \dots & b_n \\ \left| \right. & \left| \right. & \dots & \left| \right. \end{pmatrix}^T \quad (28)$$

$$= \begin{pmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_n \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \dots & a_2 \cdot b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n \cdot b_1 & a_n \cdot b_2 & \dots & a_n \cdot b_n \end{pmatrix}^T \quad (29)$$

$$= \begin{pmatrix} a_1 \cdot b_1 & a_2 \cdot b_1 & \dots & a_n \cdot b_1 \\ a_1 \cdot b_2 & a_2 \cdot b_2 & \dots & a_n \cdot b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 \cdot b_n & a_2 \cdot b_n & \dots & a_n \cdot b_n \end{pmatrix} \quad (30)$$

$$= \begin{pmatrix} \dots & b_1 & \dots \\ \dots & b_2 & \dots \\ \vdots & \vdots & \vdots \\ \dots & b_n & \dots \end{pmatrix} \begin{pmatrix} \left| \right. & \left| \right. & \dots & \left| \right. \\ a_1 & a_2 & \dots & a_n \\ \left| \right. & \left| \right. & \dots & \left| \right. \end{pmatrix} \quad (31)$$

$$= B^T A^T \quad (32)$$

where b_i, a_i are denoting row and column vectors as appropriate (should probably make this more rigorous in fact).

2.2 Conjugate

Definition given in Defn. 1.4.

Property 2.3 *Matrix conjugate is distributive over addition and scalar multiplication*

Pf.

$$\overline{A + cB} = \begin{pmatrix} \overline{a_{11} + cb_{11}} & \overline{a_{12} + cb_{12}} & \dots & \overline{a_{1n} + cb_{1n}} \\ \overline{a_{21} + cb_{21}} & \overline{a_{22} + cb_{22}} & \dots & \overline{a_{2n} + cb_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n1} + cb_{n1}} & \overline{a_{n2} + cb_{n2}} & \dots & \overline{a_{nn} + cb_{nn}} \end{pmatrix} \quad (33)$$

$$= \begin{pmatrix} \overline{a_{11} + cb_{11}} & \overline{a_{12} + cb_{12}} & \dots & \overline{a_{1n} + cb_{1n}} \\ \overline{a_{21} + cb_{21}} & \overline{a_{22} + cb_{22}} & \dots & \overline{a_{2n} + cb_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n1} + cb_{n1}} & \overline{a_{n2} + cb_{n2}} & \dots & \overline{a_{nn} + cb_{nn}} \end{pmatrix} \quad (34)$$

$$= \begin{pmatrix} \overline{a_{11}} + \overline{cb_{11}} & \overline{a_{12}} + \overline{cb_{12}} & \dots & \overline{a_{1n}} + \overline{cb_{1n}} \\ \overline{a_{21}} + \overline{cb_{21}} & \overline{a_{22}} + \overline{cb_{22}} & \dots & \overline{a_{2n}} + \overline{cb_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n1}} + \overline{cb_{n1}} & \overline{a_{n2}} + \overline{cb_{n2}} & \dots & \overline{a_{nn}} + \overline{cb_{nn}} \end{pmatrix} \quad (35)$$

$$= \overline{A} + c^* \overline{B} \quad (36)$$

where in Eq. 35 we use the fact that the complex conjugate distributes over addition and multiplication.

2.3 Hermitian Conjugate

Definition given in Defn. 1.5.

Corollary 2.4 *For $A, B \in \mathbf{M}(n, \mathbb{C})$, $c \in \mathbb{C}$, $(A + cB)^\dagger = A^\dagger + c^* B^\dagger$*

Pf.

$$(A + cB)^\dagger = \overline{(A + cB)^T} \quad (37)$$

$$= \overline{A^T + cB^T} \quad (38)$$

$$= \overline{A^T} + \overline{cB^T} \quad (39)$$

$$= A^\dagger + c^* B^\dagger \quad (40)$$

2.4 Trace

Definition given in Defn. 1.7.

Property 2.5 *Trace is distributive over addition and scalar multiplication*

Pf.

For $A, B \in \mathbf{M}(n, \mathbb{F})$,

$$\mathrm{Tr}[A + cB] = \mathrm{Tr} \left[\begin{pmatrix} a_{11} + cb_{11} & a_{12} + cb_{12} & \dots & a_{1n} + cb_{1n} \\ a_{21} + cb_{21} & a_{22} + cb_{22} & \dots & a_{2n} + cb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + cb_{n1} & a_{n2} + cb_{n2} & \dots & a_{nn} + cb_{nn} \end{pmatrix} \right] \quad (41)$$

$$= (a_{11} + cb_{11}) + (a_{22} + cb_{22}) + \dots + (a_{nn} + cb_{nn}) \quad (42)$$

$$= (a_{11} + a_{22} + \dots + a_{nn}) + c(b_{11} + b_{22} + \dots + b_{nn}) \quad (43)$$

$$= \mathrm{Tr}[A] + c\mathrm{Tr}[B] \quad (44)$$

Property 2.6 *Trace is invariant to transpose*

Pf.

$$\mathrm{Tr}[A^T] = \sum_i^n [A^T]_{ii} = \sum_i^n [A]_{ii} = \mathrm{Tr}[A] \quad (45)$$

Property 2.7 *Trace is invariant to cyclic permutation. Ie. for any matrices $A, B, C \in \mathbf{M}(n, \mathbb{F})$,*

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB] = \mathrm{Tr}[BCA] \quad (46)$$

This is known as the cyclic property of the trace and is general for any number of products.

Pf.

Consider the trace of k products. The general case of $k = n$ can be proven with just $k = 2$.

Base case ($k = 2$):

Let A, B be matrices in $\mathbf{M}(n, \mathbb{F})$ s.t. A has rows a_{i*} and columns a_{*j} . Similarly

B has rows b_{i*} and columns b_{*j} . Ie.,

$$A = \begin{pmatrix} \dots & a_{1*} & \dots \\ \dots & a_{2*} & \dots \\ & \vdots & \\ \dots & a_{n*} & \dots \end{pmatrix} = \begin{pmatrix} | & | & & | \\ a_{*1} & a_{*2} & \dots & a_{*n} \\ | & | & & | \end{pmatrix} \quad (47)$$

$$B = \begin{pmatrix} \dots & b_{1*} & \dots \\ \dots & b_{2*} & \dots \\ & \vdots & \\ \dots & b_{n*} & \dots \end{pmatrix} = \begin{pmatrix} | & | & & | \\ b_{*1} & b_{*2} & \dots & b_{*n} \\ | & | & & | \end{pmatrix} \quad (48)$$

$$(49)$$

Then,

$$\text{Tr}[AB] = \text{Tr} \left[\begin{pmatrix} \dots & a_{1*} & \dots \\ \dots & a_{2*} & \dots \\ & \vdots & \\ \dots & a_{n*} & \dots \end{pmatrix} \begin{pmatrix} | & | & & | \\ b_{*1} & b_{*2} & \dots & b_{*n} \\ | & | & & | \end{pmatrix} \right] \quad (50)$$

$$= \text{Tr} \left[\begin{pmatrix} a_{1*} \cdot b_{*1} & a_{1*} \cdot b_{*2} & \dots & a_{1*} \cdot b_{*n} \\ a_{2*} \cdot b_{*1} & a_{2*} \cdot b_{*2} & \dots & a_{2*} \cdot b_{*n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n*} \cdot b_{*1} & a_{n*} \cdot b_{*2} & \dots & a_{n*} \cdot b_{*n} \end{pmatrix} \right] \quad (51)$$

$$= \sum_i^n a_{i*} \cdot b_{*i} = \sum_i^n \sum_j^n a_{ij} b_{ji} \quad (52)$$

$$= \sum_j^n \sum_i^n b_{ji} a_{ij} = \sum_j^n b_{j*} \cdot a_{*j} \quad (53)$$

$$= \text{Tr} \left[\begin{pmatrix} b_{1*} \cdot a_{*1} & b_{1*} \cdot a_{*2} & \dots & b_{1*} \cdot a_{*n} \\ b_{2*} \cdot a_{*1} & b_{2*} \cdot a_{*2} & \dots & b_{2*} \cdot a_{*n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n*} \cdot a_{*1} & b_{n*} \cdot a_{*2} & \dots & b_{n*} \cdot a_{*n} \end{pmatrix} \right] \quad (54)$$

$$= \text{Tr} \left[\begin{pmatrix} \dots & b_{1*} & \dots \\ \dots & b_{2*} & \dots \\ & \vdots & \\ \dots & b_{n*} & \dots \end{pmatrix} \begin{pmatrix} | & | & & | \\ a_{*1} & a_{*2} & \dots & a_{*n} \\ | & | & & | \end{pmatrix} \right] \quad (55)$$

$$= \text{Tr}[BA] \quad (56)$$

Illustrative case: ($k = 3$)

$$\begin{aligned}
\text{Tr}[ABC] &= \text{Tr}[(AB)C] = \text{Tr}[C(AB)] \\
&= \text{Tr}[CAB] = \text{Tr}[(CA)B] = \text{Tr}[B(CA)] \\
&= \text{Tr}[BCA]
\end{aligned} \tag{57}$$

General case: ($k = n$)

$$\begin{aligned}
\text{Tr}[A_1 A_2 \dots A_{n-1} A_n] &= \text{Tr}[A_1 (A_2 \dots A_{n-1} A_n)] = \text{Tr}[(A_2 \dots A_{n-1} A_n) A_1] \\
&= \text{Tr}[A_2 \dots A_{n-1} A_n A_1] \\
&= \dots \\
&= \text{Tr}[A_i A_{i+1} \dots A_1 A_2 \dots A_{n-1} A_n \dots A_{i-2} A_{i-1}] \\
&= \text{Tr}[(A_i A_{i+1} \dots A_1 A_2 \dots A_{n-1} A_n \dots A_{i-2}) A_{i-1}] \\
&= \text{Tr}[A_{i-1} (A_i A_{i+1} \dots A_1 A_2 \dots A_{n-1} A_n \dots A_{i-2})] \\
&= \text{Tr}[A_{i-1} A_i A_{i+1} \dots A_1 A_2 \dots A_{n-1} A_n \dots A_{i-2}] \\
&= \dots
\end{aligned} \tag{58}$$

Corollary 2.8 *Trace is invariant to change of basis.*

Pf.

$\text{Tr}[U^\dagger A U] = \text{Tr}[U U^\dagger A] = \text{Tr}[A]$ for unitary U by Prop 2.7.

3 Symmetric matrix properties

Definition 3.1 Let $\mathbf{S}(n) \subset \mathbf{M}(n, \mathbb{R})$ denote the set of symmetric real $n \times n$ matrices (as defined in Defn. 1.8).

Property 3.2 $\mathbf{S}(n)$ is closed under addition and scalar multiplication.

Pf.

For $A, B \in \mathbf{S}(n)$,

$$(A + cB)^T = A^T + cB^T = A + cB \tag{59}$$

Property 3.3 For symmetric A, B , their product $AB \in \mathbf{S}(n) \iff AB = BA$

Pf.

$AB \in \mathbf{S}(n) \implies AB = BA$

$$AB = (AB)^T = B^T A^T = BA \tag{60}$$

$AB \in \mathbf{S}(n) \iff AB = BA$

$$(AB)^T = B^T A^T = BA = AB \tag{61}$$

Corollary 3.4 For $m \in \mathbb{Z}$, if $A \in \mathbf{S}(n)$, then $A^m \in \mathbf{S}(n)$

Pf.

Follows directly from Prop. 3.3 and the fact that a matrix commutes with itself.

4 Anti-symmetric/skew-symmetric matrix properties

Definition 4.1 Let $\mathbf{SS}(n)$ denote the set of skew-symmetric real $n \times n$ matrices (as defined in Defn. 1.8).

For the following properties, assume $A, B \in \mathbf{SS}(n)$

Property 4.2 $\mathbf{SS}(n)$ closed under addition and scalar multiplication.

Pf.

$$(A + cB)^T = A^T + cB^T = -A - cB = -(A + cB) \quad (62)$$

5 Hermitian matrices

Definition 5.1 Let $\mathbf{H}(n) \subset \mathbf{M}(n, \mathbb{C})$ denote the set of Hermitian $n \times n$ matrices (as defined in 1.9).

Property 5.2 $\mathbf{H}(n)$ closed under addition

Pf. For $A, B \in \mathbf{H}(n)$,

$$(A + B)^\dagger = A^\dagger + B^\dagger = (A + B) \quad (63)$$

by 2.4

Key result for quantum mechanics:

Property 5.3 If A is Hermitian, then A has real eigenvalues

Pf.

Let λ_i be the i^{th} eigenvalue of A with eigenvector v_i . For convenience, assume the eigenvectors are normalized to 1 ($v_i \cdot v_i = v_i^\dagger v_i = 1$). Then,

$$Av_i = \lambda_i v_i \quad (64)$$

$$v_i^\dagger Av_i = \lambda_i v_i^\dagger v_i \quad (65)$$

$$v_i^\dagger Av_i = \lambda_i \quad (66)$$

Taking the hermitian conjugate of Eq. 64,

$$v_i^\dagger A^\dagger = \lambda_i^* v_i^\dagger \quad (67)$$

$$v_i^\dagger A^\dagger v_i = \lambda_i^* v_i^\dagger v_i \quad (68)$$

$$v_i^\dagger A v_i = \lambda_i^* \quad (69)$$

Subtracting Eqs. 66 and 69,

$$0 = \lambda_i - \lambda_i^* \quad (70)$$

$$= \text{Re}[\lambda_i] + \text{Im}[\lambda_i]i - (\text{Re}[\lambda_i] - \text{Im}[\lambda_i]i) \quad (71)$$

$$= 2\text{Im}[\lambda_i] \quad (72)$$

Then each of λ_i are purely real, since $\text{Im}\lambda_i = 0$.