Useful facts about square matrices

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1 Notation/Definition cheat sheet

Definition 1.1 *Let* $\mathbf{M}(n, \mathbb{F})$ *denote the set of* $n \times n$ *matrices over field* $\mathbb{F}$.

Specifically, let $\mathbf{M}(n, \mathbb{R})$ *denote the set of real* $n \times n$ *matrices and* $\mathbf{M}(n, \mathbb{C})$ *the set of complex* $n \times n$ *matrices.*

Definition 1.2 *Let* $I_n \in \mathbf{M}(n, \mathbb{F})$ *be the matrix s.t. for all matrices* A *in* $\mathbf{M}(n, \mathbb{F})$,

$$I_n A = A I_n = A \tag{1}$$

Call this matrix the identity matrix. When n and $\mathbb{F}$ are unambiguous or arbitrary, denote the identity matrix as simply I.

1.1 Matrix Operations

Definition 1.3 *For matrix $A \in \mathbf{M}(n, \mathbb{F})$ with elements a_{ij}, let A^T be the transpose of A defined by $[A^T]_{ij} = a_{ji}$. Equivalently,*

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{2}$$

Definition 1.4 *For $A \in \mathbf{M}(n, \mathbb{C})$ with elements a_{ij}, define $\overline{A}$ as the conjugate of A defined by the map*

$$a_{ij} \mapsto a^*_{ij} \tag{3}$$

ie. take the complex conjugate of each element.

Definition 1.5 *For $A \in \mathbf{M}(n, \mathbb{C})$, define $A^\dagger$ as the Hermitian conjugate (or conjugate transpose) of A defined by*

$$A^\dagger = (\overline{A})^T = \overline{A^T} \tag{4}$$

Equivalently defined by the map

$$a_{ij} \mapsto a^*_{ji} \tag{5}$$

Definition 1.6 *For matrices $A, B \in \mathbf{M}(n, \mathbb{F})$, let $[A, B]$ be the commutator of A and B defined by*

$$[A, B] = AB - BA \tag{6}$$

Similarly, let $\{A, B\}$ be the anti–commutator of A and B defined by

$$\{A, B\} = AB + BA \tag{7}$$

We say A and B commute iff

$$[A, B] = 0 \iff AB = BA \tag{8}$$

and say A and B anti–commute iff

$$\{A, B\} = 0 \iff AB = -BA \tag{9}$$

Definition 1.7 *For matrix $A \in \mathbf{M}(n, \mathbb{F})$, let $Tr[A]$ be the trace of A defined by*

$$Tr[A] = \sum_{i}^{n} a_{ii} \tag{10}$$

ie. the sum of diagonal elements of A.

1.2 Matrix Properties

Definition 1.8 *For $A \in \mathbf{M}(n, \mathbb{R})$, say A is symmetric iff*

$$A = A^T \tag{11}$$

and say A is anti–symmetric or skew–symmetric iff

$$A = -A^T \tag{12}$$

Definition 1.9 *For $A \in \mathbf{M}(n, \mathbb{C})$, say A is Hermitian iff*

$$A = A^\dagger \tag{13}$$

and say A is anti–Hermitian or skew–Hermitian iff

$$A = -A^\dagger \tag{14}$$

Definition 1.10 *Say $A \in \mathbf{M}(n, \mathbb{F})$ with elements a_{ij} is diagonal iff $a_{ij} = 0 \; \forall \; i \neq j$. Ie. A has off-diagonal elements of only 0.*

Definition 1.11 *Say $A \in \mathbf{M}(n, \mathbb{F})$ is invertible iff $\exists$ a unique matrix A^{-1} s.t.*

$$AA^{-1} = A^{-1}A = I \tag{15}$$

If such an A^{-1} exists, call it the inverse of A.

Definition 1.12 *Say $A \in \mathbf{M}(n, \mathbb{R})$ is orthogonal iff A is invertible and*

$$A^{-1} = A^T \tag{16}$$

ie. the inverse of A is also its transpose.

Definition 1.13 *For $A \in \mathbf{M}(n, \mathbb{C})$, say A is unitary iff A is invertible and*

$$A^\dagger = A^{-1} \tag{17}$$

ie. the inverse of A is also its Hermitian conjugate.

Definition 1.14 *For $A \in \mathbf{M}(n, \mathbb{R})$, say A is diagonalizable iff $\exists$ orthogonal matrix S and diagonal matrix D s.t.*

$$A = SDS^T \tag{18}$$

For $A \in \mathbf{M}(n, \mathbb{C})$, call A diagonalizable if $\exists$ unitary S and diagonal D s.t.

$$A = SDS^\dagger \tag{19}$$

1.3 Vectors

We will need a notion of vectors and vector notation to describe some matrix properties.

Definition 1.15 *Let an $n \times 1$ matrix be called a column vector and a $1 \times n$ matrix be called a row vector.*

Definition 1.16 *For real n-dimensional column vectors u and v, define $u \cdot v$ as the dot product defined as*

$$u \cdot v = u^T v = \sum_i^n u_i v_i \tag{20}$$

where u_i, v_i are the i^{th} elements of u and v respectively.

Similarly, for complex n-dimensional u and v, define the dot product as

$$u \cdot v = u^\dagger v = \sum_i^n u_i^* v_i \tag{21}$$

Remark 1.17 *Note that the above dot product is a specific case of the inner product (denoted $\langle u, v \rangle$) which, along with a vector space, defines an inner product space.*

1.4 Basis

Definition 1.18 *Say two vectors v and u are linearly independent iff*

$$\langle v, u \rangle = 0 \tag{22}$$

Say set of vectors $\{v_1, v_2, ..., v_n\}$ is linearly independent if for all $i \neq j$, v_i and v_j are linearly independent.

Definition 1.19 *For set of vectors $V = \{v_1, v_2, ..., v_k\}$ over field F, let the span of V (denoted $Sp[V]$) be the set of linear combinations of vectors in V, ie.*

$$Sp[V] = \{c_1 v_1 + c_2 v_2 + ... + c_k v_k \mid c_i \in F\} \tag{23}$$

Definition 1.20 *For a set of vectors V, say $B = \{b_1, ..., b_k\}$ forms a basis of V iff*

1. *B is linearly independent*
2. *$Sp[B] = V$*

Additionally, define the dimension of V as the cardinality of a basis of V.

1.5 Eigenvectors/Eigenvalues

TODO

1.6 Misc.

Sometimes, it is convenient to represent a matrix by its rows or columns:

$$A = \begin{pmatrix} --- & r_1 & --- \\ --- & r_2 & --- \\ & \vdots & \\ --- & r_n & --- \end{pmatrix} = \begin{pmatrix} | & | & & | \\ | & | & & | \\ c_1 & c_2 & ... & c_n \\ | & | & & | \\ | & | & & | \end{pmatrix} \tag{24}$$

where $r_i = (a_{i1}, a_{i2}, ..., a_{in})$ and $c_j = (a_{1j}, a_{2j}, ..., a_{nj})$. Sometimes we use the notation $a_{i*} = (a_{i1}, a_{i2}, ..., a_{in})$ for rows and $a_{*j} = (a_{1j}, a_{2j}, ..., a_{nj})$ for columns.

2 Operator properties

2.1 Transpose

Definition given in Defn. 1.3. For the following, assume $A, B \in \mathbf{M}(n, \mathbb{F})$ and $c \in \mathbb{F}$

Property 2.1 *Matrix transpose is distributive over addition and scalar multiplication*

Pf.

$$(A + cB)^T = (\begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & ... & a_{nn} \end{pmatrix} + \begin{pmatrix} cb_{11} & ... & cb_{1n} \\ \vdots & \ddots & \vdots \\ cb_{n1} & ... & cb_{nn} \end{pmatrix})^T \tag{25}$$

$$= \begin{pmatrix} a_{11} + cb_{11} & ... & a_{1n} + cb_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + cb_{n1} & ... & a_{nn} + cb_{nn} \end{pmatrix}^T \tag{26}$$

$$= \begin{pmatrix} a_{11} + cb_{11} & ... & a_{n1} + cb_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + cb_{1n} & ... & a_{nn} + cb_{nn} \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} a_11 & ... & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & ... & a_{nn} \end{pmatrix} + c \begin{pmatrix} b_{11} & ... & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & ... & b_{nn} \end{pmatrix} \tag{28}$$

$$= A^T + cB^T \tag{29}$$

Property 2.2 $(AB)^T = B^T A^T$

Pf.

$$(AB)^T = (\begin{pmatrix} ... & a_1 & ... \\ ... & a_2 & ... \\ & \vdots & \\ ... & a_n & ... \end{pmatrix} \begin{pmatrix} | & | & & | \\ | & | & & | \\ b_1 & b_2 & ... & b_n \\ | & | & & | \\ | & | & & | \end{pmatrix})^T \tag{30}$$

$$= \begin{pmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & ... & a_1 \cdot b_n \\ a_2 \cdot b_1 & a_2 \cdot b_2 & ... & a_2 \cdot b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n \cdot b_1 & a_n \cdot b_2 & ... & a_n \cdot b_n \end{pmatrix}^T \tag{31}$$

$$= \begin{pmatrix} a_1 \cdot b_1 & a_2 \cdot b_1 & ... & a_n \cdot b_1 \\ a_1 \cdot b_2 & a_2 \cdot b_2 & ... & a_n \cdot b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 \cdot b_n & a_2 \cdot b_n & ... & a_n \cdot b_n \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} ... & b_1 & ... \\ ... & b_2 & ... \\ & \vdots & \\ ... & b_n & ... \end{pmatrix} \begin{pmatrix} | & | & & | \\ | & | & & | \\ a_1 & a_2 & ... & a_n \\ | & | & & | \\ | & | & & | \end{pmatrix} \tag{33}$$

$$= B^T A^T \tag{34}$$

where b_i, a_i are denoting row and column vectors as appropriate (should probably make this more rigorous in fact).

2.2 Conjugate

Definition given in Defn. 1.4.

Property 2.3 *Matrix conjugate is distributive over addition and scalar multiplication*

Pf.

$$\overline{A + cB} = \overline{\begin{pmatrix} a_{11} + cb_{11} & a_{12} + cb_{12} & \dots & a_{1n} + cb_{1n} \\ a_{21} + cb_{21} & a_{22} + cb_{22} & \dots & a_{2n} + cb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + cb_{n1} & a_{n2} + cb_{n2} & \dots & a_{nn} + cb_{nn} \end{pmatrix}} \tag{35}$$

$$= \begin{pmatrix} \overline{a_{11} + cb_{11}} & \overline{a_{12} + cb_{12}} & \dots & \overline{a_{1n} + cb_{1n}} \\ \overline{a_{21} + cb_{21}} & \overline{a_{22} + cb_{22}} & \dots & \overline{a_{2n} + cb_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n1} + cb_{n1}} & \overline{a_{n2} + cb_{n2}} & \dots & \overline{a_{nn} + cb_{nn}} \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} \overline{a_{11}} + \overline{c}\overline{b_{11}} & \overline{a_{12}} + \overline{c}\overline{b_{12}} & \dots & \overline{a_{1n}} + \overline{c}\overline{b_{1n}} \\ \overline{a_{21}} + \overline{c}\overline{b_{21}} & \overline{a_{22}} + \overline{c}\overline{b_{22}} & \dots & \overline{a_{2}n} + \overline{c}\overline{b_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n1}} + \overline{c}\overline{b_{n1}} & \overline{a_{n2}} + \overline{c}\overline{b_{n2}} & \dots & \overline{a_{nn}} + \overline{c}\overline{b_{nn}} \end{pmatrix} \tag{37}$$

$$= \overline{A} + c^* \overline{B} \tag{38}$$

where in Eq. 37 we use the fact that the complex conjugate distributes over addition and multiplication.

2.3 Hermitian Conjugate

Definition given in Defn. 1.5.

Corollary 2.4 *For $A, B \in \mathbf{M}(n, \mathbb{C})$, $c \in \mathbb{C}, (A + cB)^\dagger = A^\dagger + c^* B^\dagger$*

Pf.

$$(A + cB)^\dagger = \overline{(A + cb)^T} \tag{39}$$

$$= \overline{A^T + cB^T} \tag{40}$$

$$= \overline{A^T} + \overline{c}\overline{B^T} \tag{41}$$

$$= A^\dagger + c^* B^\dagger \tag{42}$$

2.4 Commutator

Definition in Defn. 1.6

Property 2.5 *For all matrices A, B,*

$$[A, B] = AB - BA = -(BA - AB) = -[B, A] \tag{43}$$

Property 2.6 *Commutator is a billinear form, ie.*

$$[\lambda(A_1 + A_2), B] = \lambda[A_1, B] + \lambda[A_2, B] \tag{44}$$

$$[A, \lambda(B_1 + B_2)] = \lambda[A, B_1] + \lambda[A, B_2] \tag{45}$$

Pf.

$$[\lambda(A_1 + A_2), B] = \lambda(A_1 + A_2)B - B\lambda(A_1 + A_2) \tag{46}$$
$$= \lambda(A_1B - BA_1) + \lambda(A_2B - BA_2) \tag{47}$$
$$= \lambda[A_1, B] + \lambda[A_2, B] \tag{48}$$

By Prop. 2.5, this also applies to $A, \lambda(B_1 + B_2)$.

Property 2.7 *For matrices A, B,*

$$[A^2, B] = A[A, B] + [A, B]A \tag{49}$$

In general,

$$[A^n, B] = \sum_{j=0}^{n-1} A^j[A, B]A^{n-j-1} \tag{50}$$

Pf. TODO

Similarly,

Property 2.8 *For matrices A, B, C s.t. $[A, B] = [A, C] = 0$*

$$[A, BC] = 0 \tag{51}$$

Ie. if a matrix A commutes with two matrices B, C, it also commutes with the product BC.

Pf.

$$[A, BC] = ABC - BCA \tag{52}$$
$$= ABC - BAC + BAC - BCA \tag{53}$$
$$= (AB - BA)C + B(AC - CA) \tag{54}$$
$$= [A, B]C + B[A, C] \tag{55}$$
$$= 0 \tag{56}$$

2.5 Trace

Definition given in Defn. 1.7.

Property 2.9 *Trace is distributive over addition and scalar multiplication*

Pf.

For $A, B \in \mathbf{M}(n, \mathbb{F})$,

$$\mathrm{Tr}[A + cB] = \mathrm{Tr}[\begin{pmatrix} a_{11} + cb_{11} & a_{12} + cb_{12} & ... & a_{1n} + cb_{1n} \\ a_{21} + cb_{21} & a_{22} + cb_{22} & ... & a_{2n} + cb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + cb_{n1} & a_{n2} + cb_{n2} & ... & a_{nn} + cb_{nn} \end{pmatrix}] \tag{57}$$

$$= (a_{11} + cb_{11}) + (a_{22} + cb_{22}) + ... + (a_{nn} + cb_{nn} \tag{58}$$

$$= (a_{11} + a_{22} + ... + a_{nn}) + c(b_{11} + b_{22} + ... + b_{nn}) \tag{59}$$

$$= \mathrm{Tr}[A] + c\mathrm{Tr}[B] \tag{60}$$

Property 2.10 *Trace is invariant to transpose*

Pf.

$$\mathrm{Tr}[A^T] = \sum_i^n [A^T]_{ii} = \sum_i^n [A]_{ii} = \mathrm{Tr}[A] \tag{61}$$

Property 2.11 *Trace is invariant to cyclic permutation. Ie. for any matrices $A, B, C \in \mathbf{M}(n, \mathbb{F})$,*

$$Tr[ABC] = Tr[CAB] = Tr[BCA] \tag{62}$$

This is known as the cyclic property of the trace and is general for any number of products.

Pf.

Consider the trace of k products. The general case of $k = n$ can be proven with just $k = 2$.

Base case ($k = 2$):

Let A, B be matrices in $\mathbf{M}(n, \mathbb{F})$ s.t. A has rows a_{i*} and columns a_{*j}. Similarly B has rows b_{i*} and columns b_{*j}. Ie.,

$$A = \begin{pmatrix} ... & a_{1*} & ... \\ ... & a_{2*} & ... \\ & \vdots & \\ ... & a_{n*} & ... \end{pmatrix} = \begin{pmatrix} | & | & & | \\ | & | & & | \\ a_{*1} & a_{*2} & ... & a_{*n} \\ | & | & & | \\ | & | & & | \end{pmatrix} \tag{63}$$

$$B = \begin{pmatrix} ... & b_{1*} & ... \\ ... & b_{2*} & ... \\ & \vdots & \\ ... & b_{n*} & ... \end{pmatrix} = \begin{pmatrix} | & | & & | \\ | & | & & | \\ b_{*1} & b_{*2} & ... & b_{*n} \\ | & | & & | \\ | & | & & | \end{pmatrix} \tag{64}$$

$$\tag{65}$$

Then,

$$\operatorname{Tr}[AB] = \operatorname{Tr}[\begin{pmatrix} ... & a_{1*} & ... \\ ... & a_{2*} & ... \\ & \vdots & \\ ... & a_{n*} & ... \end{pmatrix} \begin{pmatrix} | & | & & | \\ | & | & & | \\ b_{*1} & b_{*2} & ... & b_{*n} \\ | & | & & | \\ | & | & & | \end{pmatrix}] \tag{66}$$

$$= \operatorname{Tr}[\begin{pmatrix} a_{1*} \cdot b_{*1} & a_{1*} \cdot b_{*2} & ... & a_{1*} \cdot b_{*n} \\ a_{2*} \cdot b_{*1} & a_{2*} \cdot b_{*2} & ... & a_{2*} \cdot b_{*n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n*} \cdot b_{*1} & a_{n*} \cdot b_{*2} & ... & a_{n*} \cdot b_{*n} \end{pmatrix}] \tag{67}$$

$$= \sum_i^n a_{i*} \cdot b_{*i} = \sum_i^n \sum_j^n a_{ij} b_{ji} \tag{68}$$

$$= \sum_j^n \sum_i^n b_{ji} a_{ij} = \sum_j^n b_{j*} \cdot a_{*j} \tag{69}$$

$$= \operatorname{Tr}[\begin{pmatrix} b_{1*} \cdot a_{*1} & b_{1*} \cdot a_{*2} & ... & b_{1*} \cdot a_{*n} \\ b_{2*} \cdot a_{*1} & b_{2*} \cdot a_{*2} & ... & b_{2*} \cdot a_{*n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n*} \cdot a_{*1} & b_{n*} \cdot a_{*2} & ... & b_{n*} \cdot a_{*n} \end{pmatrix}] \tag{70}$$

$$= \operatorname{Tr}[\begin{pmatrix} ... & b_{1*} & ... \\ ... & b_{2*} & ... \\ & \vdots & \\ ... & b_{n*} & ... \end{pmatrix} \begin{pmatrix} | & | & & | \\ | & | & & | \\ a_{*1} & a_{*2} & ... & a_{*n} \\ | & | & & | \\ | & | & & | \end{pmatrix}] \tag{71}$$

$$= \operatorname{Tr}[BA] \tag{72}$$

Illustrative case: $(k = 3)$

$$\begin{aligned} &\operatorname{Tr}[ABC] = \operatorname{Tr}[(AB)C] = \operatorname{Tr}[C(AB)] \\ &= \operatorname{Tr}[CAB] = \operatorname{Tr}[(CA)B] = \operatorname{Tr}[B(CA)] \\ &= \operatorname{Tr}[BCA] \end{aligned} \tag{73}$$

General case: $(k = n)$

$$\begin{aligned}
\mathrm{Tr}[A_1 A_2 ... A_{n-1} A_n] &= \mathrm{Tr}[A_1 (A_2 ... A_{n-1} A_n)] = \mathrm{Tr}[(A_2 ... A_{n-1} A_n) A_1] \\
&= \mathrm{Tr}[A_2 ... A_{n-1} A_n A_1] \\
&= ... \\
&= \mathrm{Tr}[A_i A_{i+1} ... A_1 A_2 ... A_{n-1} A_n ... A_{i-2} A_{i-1}] \\
&= \mathrm{Tr}[(A_i A_{i+1} ... A_1 A_2 ... A_{n-1} A_n ... A_{i-2}) A_{i-1}] \\
&= \mathrm{Tr}[A_{i-1} (A_i A_{i+1} ... A_1 A_2 ... A_{n-1} A_n ... A_{i-2})] \\
&= \mathrm{Tr}[A_{i-1} A_i A_{i+1} ... A_1 A_2 ... A_{n-1} A_n ... A_{i-2}] \\
&= ...
\end{aligned} \tag{74}$$

Corollary 2.12 *Trace is invariant to change of basis.*

Pf.

$\mathrm{Tr}[U^\dagger A U] = \mathrm{Tr}[U U^\dagger A] = \mathrm{Tr}[A]$ for unitary U by Prop 2.11.

3 Symmetric matrix properties

Definition 3.1 *Let* $\mathbf{S}(n) \subset \mathbf{M}(n, \mathbb{R})$ *denote the set of symmetric real* $n \times n$ *matrices (as defined in Defn. 1.8).*

Property 3.2 $\mathbf{S}(n)$ *is closed under addition and scalar multiplication.*

Pf.

For $A, B \in \mathbf{S}(n)$,

$$(A + cB)^T = A^T + cB^T = A + cB \tag{75}$$

Property 3.3 *For symmetric* A, B*, their product* $AB \in \mathbf{S}(n) \iff AB = BA$

Pf.

$AB \in \mathbf{S}(n) \implies AB = BA$

$$AB = (AB)^T = B^T A^T = BA \tag{76}$$

$AB \in \mathbf{S}(n) \impliedby AB = BA$

$$(AB)^T = B^T A^T = BA = AB \tag{77}$$

Corollary 3.4 *For* $m \in \mathbb{Z}$*, if* $A \in \mathbf{S}(n)$*, then* $A^m \in \mathbf{S}(n)$

Pf.

Follows directly from Prop. 3.3 and the fact that a matrix commutes with itself.

4 Anti-symmetric/skew-symmetric matrix properties

Definition 4.1 *Let* $\mathbf{SS}(n)$ *denote the set of skew-symmetric real* $n \times n$ *matrices (as defined in Defn. 1.8.*

For the following properties, assume $A, B \in \mathbf{SS}(n)$

Property 4.2 $\mathbf{SS}(n)$ *closed under addition and scalar multiplication.*

Pf.

$$(A + cB)^T = A^T + cB^T = -A - cB = -(A + cB) \tag{78}$$

5 Orthogonal matrix properties

TODO

6 Hermitian matrices

Definition 6.1 *Let* $\mathbf{H}(n) \subset \mathbf{M}(n, \mathbb{C})$ *denote the set of Hermitian* $n \times n$ *matrices (as defined in 1.9).*

Property 6.2 $\mathbf{H}(n)$ *closed under addition*

Pf. For $A, B \in \mathbf{H}(n)$,

$$(A + B)^\dagger = A^\dagger + B^\dagger = (A + B) \tag{79}$$

by 2.4

Key result for quantum mechanics:

Property 6.3 *If* A *is Hermitian, then* A *has real eigenvalues*

Pf.

Let λ_i be the i^{th} eigenvalue of A with eigenvector v_i. For convenience, assume the eigenvectors are normalized to 1 ($v_i \cdot v_i = v_i^\dagger v_i = 1$). Then,

$$Av_i = \lambda_i v_i \tag{80}$$

$$v_i^\dagger A v_i = \lambda_i v_i^\dagger v_i \tag{81}$$

$$v_i^\dagger A v_i = \lambda_i \tag{82}$$

Taking the hermitian conjugate of Eq. 80,

$$v_i^\dagger A^\dagger = \lambda_i^* v_i^\dagger \tag{83}$$

$$v_i^\dagger A^\dagger v_i = \lambda_i^* v_i^\dagger v_i \tag{84}$$

$$v_i^\dagger A v_i = \lambda_i^* \tag{85}$$

Subtracting Eqs. 82 and 85,

$$0 = \lambda_i - \lambda_i^* \tag{86}$$

$$= \mathrm{Re}[\lambda_i] + \mathrm{Im}[\lambda_i]i - (\mathrm{Re}[\lambda_i] - \mathrm{Im}[\lambda_i]i) \tag{87}$$

$$= 2\mathrm{Im}[\lambda_i] \tag{88}$$

Then each of λ_i are purely real, since $\mathrm{Im}\lambda_i = 0$.

Property 6.4 *If A, B are two Hermitian matrices, then the commutator $[A, B]$ of A and B is skew-Hermitian*

Pf.

$$[A, B] = AB - BA \tag{89}$$

$$= (BA)^\dagger - (AB)^\dagger \tag{90}$$

$$= (BA - AB)^\dagger \tag{91}$$

$$= [B, A]^\dagger = -[A, B]^\dagger \tag{92}$$

where we've used the trivial property of the commutator that for any matrices A, B, the following is true: $[A, B] = AB - BA = -(BA - AB) = -[B, A]$.